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Maximally entangled states for qubit-qutrit systems

Safa Jami* and Mohsen Sarbishei

Department of Physics, Faculty of Science, Ferdowsi University of Mashhad, Iran

E-mail : sjami@sciencel.um.ac.ir

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Abstract : Entanglement provides a powerful physical resource for new kind of communication protocols and computation. Pairs with higher degree of entanglement are much better resource for this purpose. So obtaining the maximally entangled state for quantum systems is helpful. The maximally entangled states are clear in systems with equal dimension; but in the case two subsystems having different dimensions no much work has been done. In this paper, we consider qubit-qutrit system and obtain its maximally entangled states.

Keywords : Entanglement, shmidt decomposition, maximally entangled states.

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1. Introduction

Entanglement is the most surprising nonclassical property of composite quantum systems [1]. Now a days, there is a growing interest in studying entanglement due to its potential applications in quantum computing and quantum information processing [2] such as quantum cryptography [3], dense coding [4] and teleportation [5]. In order to be a well-defined characteristic, entanglement has to be quantifiable. Pairs with a high degree of entanglement, should be a better resource than less entangled ones. There are several ways to define the degree of entanglement [6–8] which have been applied for qubit-qubit systems; and in some cases, have been generalized to $N \times N$ systems. The maximally entangled states of are of special interest. These states have been obtained for two bipartite systems with equal dimensions. In this paper, we derive the maximally entangled states, Bell type states, for a qubit-qutrit system.

For this purpose, first in Section 2, we find the Schmidt coefficients of a qubit-qutrit system, using the Schmidt decomposition. Then in Section 3, we derive the maximally entangled states for this system. And finally in Section 4, we compare our results with some known entanglement measures.

2. Schmidt coefficients for a qubit-qutrit system

The density matrix for a quantum system in a pure state $|\psi\rangle$ is :

$$\rho = |\psi\rangle\langle\psi| . \tag{1}$$

If this system is composed of a qubit and a qutrit, then the state can be written in the form

$$|\psi\rangle = \sum_{i=0}^{1} \sum_{j=0}^{2} a_{ij} \, |i\rangle |j\rangle , \tag{2}$$

where $|i\rangle$ and $|j\rangle$ are orthonormal basis of the qubit and qutrit respectively. The coefficients a_{ij} satisfy the normalization relation :

$$\sum_{i,j} \qquad = \langle\psi|\psi\rangle = 1 . \tag{3}$$

According to the Schmidt decomposition theorem [2], for any $|\psi\rangle$, eq. (2) can be replaced by

$$|\psi\rangle = \sum k_\mu |x_\mu\rangle |y_\mu\rangle , \tag{4}$$

*Corresponding Author

where $\left|x_\mu\right\rangle$ and $\left|y_\mu\right\rangle$ are orthonormal linear combinations of the $|i\rangle$ and $|j\rangle$ respectively; and the quantities k_μ, Schmidt coefficients, are non-negative real numbers in the range of 0 to 1. Since there are at most two orthonormal $\left|x_\mu\right\rangle$, there are at most two non-vanishing k_μ in the Schmidt expansion (4) for $|\psi\rangle$, although there can be three orthonormal $\left|y_\mu\right\rangle$ [9]. Correspondingly, the normalization relation (3) reduces to

$$\sum_{\mu=1}^{2}(k_\mu)^2 = 1. \tag{5}$$

We know that k_μ^2 are the non-vanishing eigenvalues of the hermitian matrices $AA^\dagger$ and $A^\dagger A$, where A is :

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The corresponding set of eigenvectors are $\left|x_\mu\right\rangle$ and $\left|y_\mu\right\rangle$ respectively [10].

k_μ^2's are also the eigenvalues of the reduced density matrices ρ_1 and ρ_2 defined as [2]

$$\rho_1 = \mathrm{Tr}_2(|\psi\rangle\langle\psi|),\ \rho_2 = \mathrm{Tr}_1(|\psi\rangle\langle\psi|), \tag{7}$$

where Tr_1 and Tr_2 stand for tracing over the subspaces 1 and 2 respectively.

3. Derivation of maximally entangled states for qubit-qutrit system

For a pair of qubits, there exists a degree of entanglement that is based on Schmidt decomposition in the form $P = 2\,k_1\,k_2$, where P is the degree of entanglement, and k_1 and k_2 are the Schmidt coefficients [8]. The same relation also holds for qubit-qutrit case [9]. To derive the maximally entangled states of a qubit-qutrit system, let us consider the general state

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{02}|02\rangle + a_{10}|10\rangle + a_{11}|11\rangle + a_{12}|12\rangle. \tag{8}$$

This state may also be written in terms of the Schmidt decomposition

$$|\psi\rangle = k_1\,|x_1, y_1\rangle + k_2\,|x_2, y_2\rangle, \tag{9}$$

where

$$\begin{aligned} k_1 = \Big\{\frac{1}{2} + \frac{1}{2}\Big[1 - 4\Big(&|a_{00}|^2|a_{11}|^2 + |a_{00}|^2|a_{12}|^2 \\ &+ |a_{01}|^2|a_{10}|^2 + |a_{01}|^2|a_{12}|^2 + |a_{02}|^2|a_{10}|^2 \\ &+ |a_{02}|^2|a_{11}|^2 - a_{00}\,a_{11}\,a_{10}^*\,a_{01}^* \\ &- a_{00}\,a_{12}\,a_{10}^*\,a_{02}^* - a_{01}\,a_{10}\,a_{11}^*\,a_{00}^* \\ &- a_{01}\,a_{12}\,a_{11}\,a_{02} - a_{02}\,a_{10}\,a_{12}\,a_{10} \\ &- a_{02}\,a_{11}\,a_{12}^*\,a_{01}^*\Big)\Big]^{1/2}\Big\}^{1/2}, \end{aligned} \tag{10a}$$

$$\begin{aligned} k_2 = \Big\{\frac{1}{2} - \frac{1}{2}\Big[1 - 4\Big(&|a_{00}|^2|a_{11}|^2 + |a_{00}|^2|a_{12}| \\ &+ |a_{01}|^2|a_{10}|^2 + |a_{01}|^2|a_{12}|^2 + |a_{02}|^2|a_{10}|^2 \\ &+ |a_{02}|^2|a_{11}|^2 - a_{00}\,a_{11}\,a_{10}^*\,a_{01}^* \\ &- a_{00}\,a_{12}\,a_{10}\,a_{02} - a_{01}\,a_{10}\,a_{11}\,a_{00} \\ &- a_{01}\,a_{12}\,a_{11}^*\,a_{02}^* - a_{02}\,a_{10}\,a_{12}^*\,a_{10}^* \\ &- a_{02}\,a_{11}\,a_{12}^*\,a_{01}^*\Big)\Big]^{1/2}\Big\}^{1/2}. \end{aligned} \tag{10b}$$

Therefore, for the degree of entanglement we have :

$$\begin{aligned} P = 2\Big[&|a_{00}|^2|a_{11}|^2 + |a_{00}|^2|a_{12}|^2 + |a_{01}|^2|a_{10}|^2 \\ &+ |a_{01}|^2|a_{12}|^2 + |a_{02}|^2|a_{10}|^2 + |a_{02}|^2|a_{11}|^2 \\ &- a_{00}\,a_{11}\,a_{10}^*\,a_{01}^* - a_{00}\,a_{12}\,a_{10}^*\,a_{02}^* \\ &- a_{01}\,a_{10}\,a_{11}^*\,a_{00}^* - a_{01}\,a_{12}\,a_{11}^*\,a_{02}^* \\ &- a_{02}\,a_{10}\,a_{12}^*\,a_{10}^* - a_{02}\,a_{11}\,a_{12}^*\,a_{01}^*\Big]^{1/2}. \end{aligned} \tag{11}$$

Now, we maximize this expression with the normalization condition eq. (3) as a constraint. We obtain the following six independent states as the maximally entangled states with $P = 1$:

$$\left|\psi^\pm\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |12\rangle),$$

$$\left|\varphi^\pm\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle),$$

$$\left|\chi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|02\rangle \pm |11\rangle). \tag{12}$$

So these states are Bell type states for qubit-qutrit system.

As we mentioned before, the Schmidt coefficient can also be obtained from reduced density matrices ρ_1 and ρ_2. In the case of two subsystems with equal dimensions, for the maximally entangled state, we have :

$$\mathrm{Tr}_1(|\psi\rangle\langle\psi|) = \frac{1}{N} I_2,$$

$$\mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \frac{1}{N} I_1. \tag{13}$$

But in our case where the dimensions of two subsystems are not equal, the above relation reduces to :

$$\mathrm{Tr}_1(|\psi\rangle\langle\psi|) = \frac{1}{2} I_2',$$

$$\mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \frac{1}{2} I_1. \tag{14}$$

where, I_2' is a 3 × 3 diagonal matrix with diagonal elements (0, 1, 1). Since one of the eigenvalues of ρ_1, the density matrix which corresponds to qutrit, is always zero.

4. Comparison with other entanglement measure

In this section, we find the degree of entanglement of our Bell-type .states by some other entanglement measures.

4.1. von Neumann entropy :

For any bipartite pure state, Bennett *et al* [11] have shown that it is reasonable to define the entanglement of the system as the von Neumann entropy of either of its two parts. That is, if $|\psi\rangle$ is the state of the whole system, the entanglement can be defined as :

$$E(\psi) = -\mathrm{Tr}(\rho_1 \log_2 \rho_1) = -\mathrm{Tr}(\rho_2 \log_2 \rho_2), \tag{15}$$

where ρ_1 and ρ_2 are reduced density matrices. If λ_x are the eigenvalues of ρ_1 and ρ_2, then eq. (15) can be re-expressed as

$$E(\psi) = -\sum_x \lambda_x \log_2 \lambda_x. \tag{16}$$

Now for the states (12), we have : $\lambda_1 = \lambda_2 = \frac{1}{2}$. Therefore, $E = 1$ and the entanglement is maximized.

4.2. Entanglement of formation :

Another entanglement measure is the entanglement of formation that expresses which, for a pure state $|\psi\rangle$ which is defined as [6] :

$$E(\psi) = \varepsilon(C(\psi)), \tag{17}$$

where ε is defined as

$$\varepsilon(x) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2}\right) \tag{18}$$

where H is the binary entropy function, $H(x) = -[x \log_2 x + (1-x)\log_2(1-x)]$. The quantity C, like E for this system, ranges from zero to one, and is a kind of measure of entanglement in its own right, called concurrence.

For the states (12), we get $C = 1$ and therefore, all of these states have $E = 1$.

4.3. Negativity :

One simple measure of entanglement is based on the Peres criterion for separability : a bipartite state ρ is separable only if the partial transpose of ρ, that is, the result of applying the transpose operation to only one of the two subsystems, has no negative eigenvalues [12]. For 2 × 2 and 2 × 3 systems, this condition is not only necessary but also sufficient for separability [13]. So if ν is the smallest eigenvalue of the partial transpose of ρ, we can take $N(\rho) = 2\max\{0, -\nu\}$, called the negativity, to be a measure of ρ's entanglement [14].

For the states (12), we get $\nu = -1/2$, therefore for these states, $N(\rho) = 1$.

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